A STEADY-STATE MOVING CRACK OF LONGITUDINAL SHEAR WITH AN INFINITELY NARROW PLASTIC ZONE

P. A. Martynyuk and E. N. Sher

UDC 539.374

A simple model of longitudinal shear with a plastic zone at the tip of the crack is considered in the paper. The plasticity zone is assumed to be narrow, and fulfillment of the Mises condition is stipulated on its boundary. The crack moves with a constant velocity without change in the length.

When a crack propagates in a preliminarily deformed elastic body, a part of the elastic energy of the body is irreversibly converted into heat energy produced in a plasticity zone at the crack tip and, possibly, goes into some other energy losses that accompany fracture. If the fracture takes place in a quasi-brittle manner, i.e., material everywhere except in a small region of the tip of the crack behaves elastically, then we can imagine that the crack tip is a kind of source of elastic energy whose intensity is determined by the velocity of the crack and the singularity coefficients of the stress field, according to the known law [1, 2] for an elastic body with a moving cut.

According to this law the amount of energy per unit area of the crack, emerging irreversibly from the elastic body into the vicinity of the crack tip, is increased with an increase in the velocity of the crack and the singularity coefficients of the stress field.

Experiments with photoelastic materials [3, 4] showed that if, for example, we subject a rectangular plate, containing some initial crack, to tension with an increasing force, then at a certain critical stress the crack begins to move with increasing velocity and tears through the entire testpiece. Here the singularity coefficient of stresses turns out to be a function which monotonically grows with the length of the crack.

From what has been said it follows that the amount of energy per unit area of the crack also grows monotonically; here the specific energy expenditure can increase 2-3 times. A model with a constant surface energy is not satisfactory here. In the given investigation we consider a simple model of longitudinal shear with a plastic zone at the tip of a crack which moves with a steady state. The plasticity zone is assumed to be narrow; fulfillment of the Mises condition is stipulated on its boundary. The formulation and solution of such a problem in a static case is given in [5]. The crack moves also as assumed in [6], with a constant velocity v, without a change in the length.

We consider the motion of such a crack of length $2x_0$ in an infinite elastic body which is subjected at infinity to the stress $\tau_{yz}=\tau_\infty$. The only nonzero displacement – the displacement about the z axis – satisfies the equation

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \tag{1}$$

Here c is the velocity of transverse waves. The components of the stress tensor are zero, except τ_{yz} and τ_{xz} which are given by the relationships

$$\tau_{yz}=\mu\frac{\partial w}{\partial y},\quad \tau_{xz}=\mu\frac{\partial w}{\partial x} \tag{2}$$

where μ is the shear modulus.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Teknicheskoi Fiziki, No. 2, pp. 121-126, March-April, 1973. Original article submitted July 6, 1972.

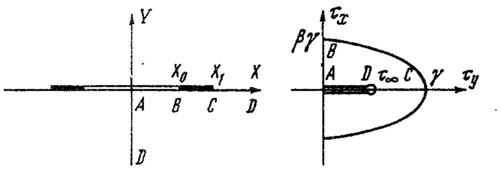

Fig. 1

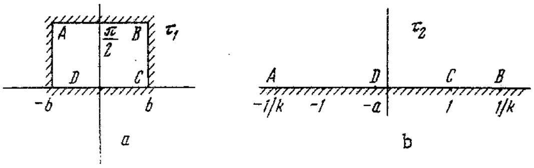

Fig. 2

Let the crack move along the straight line $y=0$. We make the following substitution of variables:

$$X = x - vt, \quad Y = \beta y, \quad \beta = (1 - v^2 / c^2)^{1/2}$$

Then Eqs. (1) and (2) assume the form

$$\frac{\partial^2 w}{\partial X^2} + \frac{\partial^2 w}{\partial Y^2} = 0 \quad ,$$

$$\tau_{yz} = \mu \frac{\partial w}{dY} \beta, \qquad \tau_{xz} = \mu \frac{\partial w}{\partial X}$$

It can be shown that the complex stress function

$$\tau = \tau_y + i\tau_x \qquad (\tau_y = \tau_{yz}, \quad \tau_x = \beta\tau_{xz})$$

is an analytic function of the complex variable $\zeta = X + iY$. In view of symmetry we can confine ourselves to the consideration of the region $\operatorname{Re}\zeta > 0$. Just as in [7-9] we assume that the plastic zones are infinitely narrow and constitute segments of the X axis so that $X_0 < X < X_1$, $Y=0$. In addition to this, we assume that at the boundary of the plastic zones the condition of Mises

$$\sqrt{\tau_{yz}^2 + \tau_{xz}^2} = \gamma \tag{3}$$

is fulfilled.

Here γ is the yield point; consequently, in the entire elastic region the condition $(\tau_{yz}{}^2 + \tau_{xz}{}^2)^{1/2} < \gamma$ must be fulfilled. The region of variation of τ, corresponding to the region $\operatorname{Re}\xi > 0$, is the interior of a half ellipse with the semiaxes γ and $\beta\gamma$ and a cut along the real axis from 0 to $\tau = \tau_\infty$ (Fig. 1).

The first quadrant of the ellipse $\operatorname{Re}\tau > 0$, $\operatorname{Im}\tau > 0$ corresponds to the region ζ

$$\operatorname{Re}\zeta > 0, \quad \operatorname{Im}\zeta < 0$$

We introduce a function $\zeta = \zeta(\tau)$ into the analysis; we have the following boundary value problem for it:

$$\operatorname{Im}\zeta = 0 \quad \text{for} \quad \begin{cases} \operatorname{Re}\tau = 0, & 0 < \operatorname{Im}\tau \leqslant \beta\gamma \\ \sqrt{\tau_y^2 + \tau_x^2/\beta^2} = \gamma, & 0 < \arg\tau < \pi/2 \\ \operatorname{Im}\tau = 0, & \tau_\infty < \operatorname{Re}\tau < \gamma \end{cases}$$

$$\operatorname{Re}\zeta = 0 \quad \text{for} \quad \operatorname{Im}\tau = 0, \quad 0 \leqslant \operatorname{Re}\tau < \tau_\infty \tag{4}$$

The function

$$\tau_1 = \ln \frac{\tau/\gamma + (\tau^2/\gamma^2 - v^2/c^2)^{1/2}}{v/c} - \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)^{1/2}$$

realizes a conformal mapping of the interior of the upper quadrant of the ellipse onto the interior of a rectangle with vertices at the points $(-b, 0)$, $(+b, 0)$, where $b = {}^1/_2 \ln[(1+\beta)/(1-\beta)]^{1/2}$, and the height $\pi/2$ (Fig. 2a). The function

$$\tau_1 = C\int_0^{\tau_2} \frac{d\tau_2}{\sqrt{(1-\tau_2^2)(1-k^2\tau_2^2)}} \tag{5}$$

gives a conformal mapping of the upper half-plane $\operatorname{Im}\tau_2 > 0$ onto the interior of the rectangle in the region τ_1, where $\tau_2 = \pm 1$ is transformed into $\tau_1 = \pm b$, while the points $\tau_2 = \pm k^{-1}$ are transformed respectively into $\tau_1 = \pm b + i\pi/2$ (Fig. 2b). The function, reciprocal to (5)

$$\tau_2 = \operatorname{sn}(\tau_1 C^{-1}, \ k) \tag{6}$$

gives a conformal mapping of the interior of the rectangle in the region τ_1 onto the upper half-plane of the region τ_2.

The parameters C and k in (5) are determined from the equations [10]

$$b = \frac{1}{2}\ln\sqrt{\frac{1+\beta}{1-\beta}} = C\int_0^1 \frac{d\tau_2}{\sqrt{(1-\tau_2^2)(1-k^2\tau_2^2)}} = CK(k^2)$$

$$\frac{\pi}{2} = C\int_1^{1/k} \frac{d\tau_2}{\sqrt{(\tau_2^2-1)(1-k^2\tau_2^2)}} = CK(k'^2) \tag{7}$$

From a simultaneous solution of these equations for a given velocity v/c, we find C and k, and with this the mapping is completely defined.

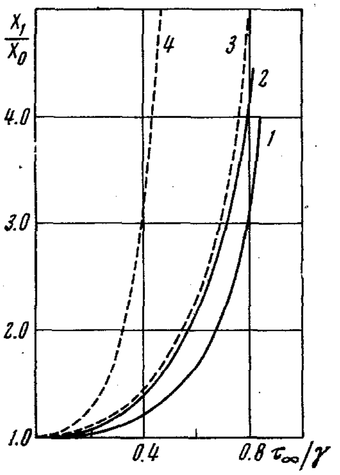

Fig. 3

In the case of the mapping (6) the point $\tau = 0$ is transformed into $\tau_2 = -1/k$, while $\tau = \tau_\infty$ is transformed into $\tau_2 = a$, which is located in the interval from $-1/k$ to $+1$, when τ_∞ varies from 0 to γ. The problem (4) for the function $\zeta = \zeta(\tau)$ in the region τ_2 is brought to the mixed boundary value problem of Keldysh–Sedov (Fig. 2b)

$$\operatorname{Im}\zeta = 0 \quad \text{for} \begin{cases} -\infty < \operatorname{Re}\tau_2 < -1/k, & \operatorname{Im}\tau_2 = 0 \\ a < \operatorname{Re}\tau_2 < +\infty, & \operatorname{Im}\tau_2 = 0 \end{cases}$$
$$\operatorname{Re}\zeta = 0 \quad \text{for} \quad -1/k < \operatorname{Re}\tau_2 < a, \qquad \operatorname{Im}\tau_2 = 0$$

The general solution of this problem is

$$\zeta = d\sqrt{\frac{\tau_2 + 1/k}{\tau_2 \pm a}}$$

The upper sign is taken in the case where $-1/k < a < 0$, while the lower sign is taken where $0 < a < 1$. The constant d is determined from the condition that $\zeta = X_0$ for $\tau = i\beta\gamma$ (or in the τ_2 plane $\tau_2 = 1/k$ corresponds to it). Hence

$$\zeta = X_0\sqrt{\frac{1 \pm ak}{2}}\sqrt{\frac{\tau_2 + 1/k}{\tau_2 \pm a}} \tag{8}$$

The quantity X_1, which determines the position of the end of the plastic zone, is found from the condition $\zeta = X_1$ for $\tau = \gamma$ (or in the τ_2 region for $\tau_2 = 1$), and finally we can write

$$\frac{X_1}{X_0} = \sqrt{\frac{(1 \pm ak)(1 + k)}{2k(1 \pm a)}} \tag{9}$$

In particular, for $v/c = \tau_\infty/\gamma$ we have $X/X_0 = (1 + k)/2\sqrt{k}$.

The available tables do not allow us to find C and k from the expression (7) for $v/c > 0.6$. For this we consider the limiting case $v/c = \alpha \to 1$.

Here, from Eqs. (7), using asymptotic representations of elliptic integrals, we obtain

$$k \approx 4\exp -\frac{\pi^2}{2\beta}, \qquad C \approx \frac{\sqrt{1 - \alpha^2}}{\pi}$$

If at the same time $v/C < \tau_\infty/\gamma$, then the point a is located in the interval $(-1/k, -1)$. From (5) we obtain an expression for determining a in the form [11]

$$-\frac{\pi}{2} + \operatorname{arc\,tg}\frac{\sqrt{\alpha^2 - \tau_\infty^2/\gamma^2}}{\tau_\infty/\gamma} = -C\int_{-1/k}^{-a}\frac{d\tau_2}{\sqrt{(\tau_2^2 - 1)(1 - k^2\tau_2^2)}} \tag{10}$$

In the approximation being considered, this equation is represented in the form

$$\operatorname{arc\,tg}\frac{\sqrt{\alpha^2 - \tau_\infty^2/\gamma^2}}{\tau_\infty/\gamma^2} \approx \frac{\sqrt{1 - \alpha^2}}{\pi}\left[\ln 2 + \operatorname{Arch} a - \frac{1}{4} + \frac{k^2}{4}a\sqrt{a^2 - 1}\right]$$
$$(1 < a < 1/k) \tag{11}$$

In Fig. 3 we have represented the results of the calculation according to the expression (9) by dashed lines (curve 3 corresponds to $v/c = 0.5$, while curve 4 corresponds to $v/c = 0.9$). The curve corresponding to $v/c = 0.1$ practically coincides with the static curve 2 of the investigation [5].

$$\frac{x_1}{x_0} = \frac{\gamma^2 + \tau_\infty^2}{\gamma^2 - \tau_\infty^2}$$

On this graph curve 1 depicts the solution [8] obtained with the condition that τ_{yz} is constant in the plasticity zone

$$\frac{x_1}{x_0} = \left[\cos\frac{\pi\tau_\infty}{2\gamma}\right]^{-1}$$

A typical distribution of τ_{yz}/γ and τ_{xz}/γ along the length of the plastic zone, and also the dimensionless displacement $\mu w/\gamma X_0$ along the length of the plastic zone, are depicted in Fig. 4 for $v/c = 0.5$ and $\tau/\gamma = 0.8$.

The relationship of the dimensionless displacement $\mu w/\gamma X_0$ at the end of the plasticity zone, i.e., at the point $\zeta = X_0$, dependent on τ_∞/γ for $v/c = 0.0$, 0.5, 0.9 is depicted in Fig. 5 by curves 1, 2, 3, respectively.

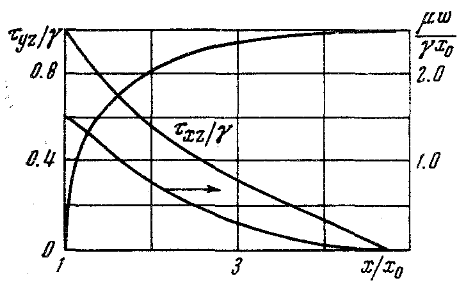

Fig. 4

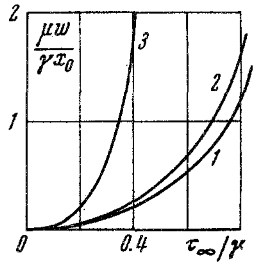

Fig. 5

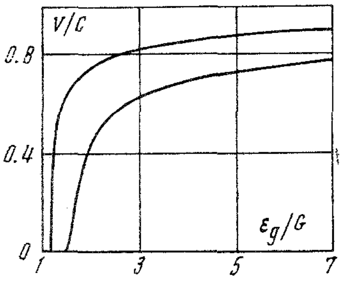

Fig. 6

For the formulation used in the investigation, the energy of the elastic body, when the crack develops, is transformed into plastic work. We determine the amount of plastic work per unit length of the crack during its motion by the expression

$$\varepsilon_g = 2 \int_{X_0}^{X_1} \tau_y \frac{\partial w}{\partial X} dX \tag{12}$$

In the case of a brittle fracture, G, the intensity of liberation of elastic energy per unit length, as follows from [12], equals

$$G = \lim_{\alpha \to 0} \frac{2}{\alpha} \int_0^\alpha \tau_y \frac{w}{2} dx = \frac{\tau_\infty^2 \pi X_0}{2\mu}$$

for a crack of length $2X_0$ and the stress $\tau_{yz} = \tau_\infty$ at infinity. In the static case, using the solution of [5] and the expression (12), we obtain

$$\varepsilon_c = \frac{2}{\mu} \frac{(\gamma^4 - \tau_\infty^4)}{4\tau_\infty^2} \int_{X_0}^{X_1} \frac{\sqrt{(X^2 - X_0^2)(X_1^2 - X^2)}}{X^2} dX$$

We note that in the case where the length of the plastic zone is small, i.e., when $\tau_\infty/\gamma \ll 1$, this expression with accuracy up to terms of the order τ_∞^2/γ^2 has the form

$$\varepsilon_c = \tau_\infty^2 \pi X_0 / 2\mu$$

which coincides with the results obtained from the theory of brittle fracture.

For $\tau_\infty/\gamma = 0.4$, 0.8 in Fig. 6 we have depicted the dependence of ε_g/G on the value v/c. The upper curve corresponds to the value $\tau_\infty/\gamma = 0.4$.

The graph just presented shows that indeed the amount of energy expended in plastic work increases as the velocity of the crack grows and the stress at infinity increases. Also the dimension of the plastic zone according to Fig. 3 and the displacement at the crack tip $X = X_0$ (Fig. 5) vary. Hence, for example, it follows that if in the role of criterion of fracture we take the displacement at the crack tip, then the critical value of such a displacement must depend on the velocity of the crack and the loading parameters.

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